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THE GENERALIZED HOMOLOGICAL CAUCHY INTEGRAL FORMULA FOR $\mathcal{A}$ -HOLOMORPHIC FUNCTIONS

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Communicated by V. Drensky

*Dedicated to the memory of the late
Corresponding Member of BAS E. Horozov*

ABSTRACT. Given a finite-dimensional commutative associative unital $\mathbb{C}$ -algebra $\mathcal{A}$ and $U \subseteq \mathcal{A}$ open (in the underlying analytic topology), a (totally) differentiable function $f: U \rightarrow \mathcal{A}$ is called $\mathcal{A}$ -holomorphic if Df is also $\mathcal{A}$ -linear. For the local case $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ we prove for this class of functions an analogue of the homological Cauchy Integral Formula, substantially strengthening a previous result of G. B. Rizza ([10], [11]). On the way to do so, we give a complete characterization of the corresponding analogue of the index (winding number) of (singular) 1-cycles in U , thus improving on previous partial results and calculations of J. Edenhofer ([4]) and V. Shpakivskyi ([13]). We also prove (maximal) $\mathcal{A}$ -analytic continuation of $\mathcal{A}$ -holomorphic functions, thus in particular characterizing the domains of $\mathcal{A}$ -holomorphy.

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Notations and conventions. Given a direct sum decomposition, say $V \oplus W$, we will write $v \oplus w$ to emphasize that the sum $v + w$ is in fact with respect to said decomposition, in particular $v \in V$ and $w \in W$.

By abuse of notation, we will use the same symbol to denote both a (singular) 1-cycle, say $\Gamma \in Z_1(U, G)$, and its support. Same goes for curves $\gamma : [0, 1] \rightarrow X$.

Ab: the category of abelian groups.

$Z_1(U, G)$: the abelian group of singular 1-cycles on U with coefficients in $G \in \mathbf{Ab}$.

$B_1(U, G)$: the abelian subgroup of $Z_1(U, G)$ of singular 1-boundaries on U with coefficients in $G \in \mathbf{Ab}$.

$\mathbb{K}$: the field $\mathbb{R}$ or $\mathbb{C}$.

$\mathcal{A}$: a commutative associative $\mathbb{K}$ -algebra.

$\mathcal{A}^\times$: the group of units (multiplicative group) of $\mathcal{A}$.

$\text{Sing}(\mathcal{A}) := \mathcal{A} \setminus \mathcal{A}^\times$: the singular locus of $\mathcal{A}$.

$:=$: defining equality.

$\stackrel{\text{def.}}{=}$: equality by definition, i.e. equality that is immediate from the definitions.

$\mathbb{D}_r(z_0)$: the open disc with radius r around $z_0 \in \mathbb{C}$.

$\mathbb{D}_r^*(z_0)$: the punctured open disc with radius r around $z_0 \in \mathbb{C}$.

$\Delta_r(Z)$: open polydisc of radius r around $Z \in \mathbb{K}^n$.

$\mathcal{C}^k(X, Y)$: space of k -times continuously differentiable maps $X \rightarrow Y$.

$\mathcal{C}_{\text{pw}}^k(X, Y)$: space of *piece-wise* k -times continuously differentiable maps $X \rightarrow Y$.

$\|\cdot\|_{\mathcal{A}}$: an *algebra* norm on $\mathcal{A}$, i.e. a submultiplicative *normalized* ($\|1_{\mathcal{A}}\|_{\mathcal{A}} = 1$) norm on $\mathcal{A}$.

$\|f\|_K := \sup_{x \in K} \|f(x)\|$: uniform norm with respect to some given subset K and some given norm $\|\cdot\|$, for example $\|\cdot\|_{\mathcal{A}, K}$.

$\Omega_{\text{hol}}^1(U, \mathcal{A})$: holomorphic 1-forms on U with values in $\mathcal{A}$.

$\mathcal{LX} := \mathcal{C}_{\text{pw}}^1(\mathbb{S}^1, X)$: free loop space of loops of regularity $\mathcal{C}_{\text{pw}}^1$.

$\text{ind}(\gamma, z_0)$: index (winding number) of a closed curve γ with respect to $z_0 \in \mathbb{C} \setminus \gamma$.

0. Introduction and overview. Let $\mathcal{A}$ be a finite-dimensional commutative associative $\mathbb{K}$ -algebra, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A (totally) differentiable function $f: U \rightarrow \mathcal{A}$, $U \subseteq \mathbb{C}$ open, is called $\mathcal{A}$ -differentiable if Df respects the $\mathcal{A}$ -structure (i.e. it is $\mathcal{A}$ -linear). The first to consider this class of functions was Scheffers [12], who stated the corresponding analogue of the Cauchy-Riemann

equations. The first more detailed treatment in the case $\mathbb{K} = \mathbb{C}$ was given by Ketchum in [7]. Later the basic theory was generalized to infinite-dimensional Banach $\mathbb{C}$ -algebras $\mathcal{A}$ by [8], then followed by [3]. The latter is still an active area of research.

In the present paper we focus on the finite-dimensional case, more specifically on the case $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$. This is not much of a restriction since every finite-dimensional commutative associative unital $\mathbb{C}$ -algebra is a finite product of local ones by the basic structure theory of Artin rings and it can be shown that the $\mathcal{A}$ -equivariant condition on Df behaves reasonably well with respect to algebra epimorphisms (in particular projections) as well as other categorical notions and constructions. But as just alluded the general case is best left for a treatment within a proper categorical setting and all the additional notation load can obscure the basic ideas, which on the other hand are very transparent in the local case.

More concretely, the main goal of the present paper is to prove an analogue of the homological Cauchy Integral Formula in the setting of $\mathcal{A}$ -holomorphic functions, topologically strengthening an old result by Rizza (see [10], [11]). Roughly speaking, Rizza proves the Cauchy Integral Formula over (singular) 1-cycles Γ that are null-homological, whereas we show that it is sufficient for only a certain projection of Γ , namely $\chi_{\#}\Gamma$, where $\chi: \mathcal{A} \rightarrow \mathcal{A}/\mathfrak{m} \cong \mathbb{C}$ is the unique character of $\mathcal{A}$, to be null-homological. This is a substantially weaker topological requirement since generically null-homology of a cycle cannot be merely deduced from null-homology of its coordinate projections (of course $[\Gamma] = 0$ implies $[\chi_{\#}\Gamma] = 0$, but certainly not vice versa). This is done in Section 5. To do so, we also prove there a (maximal) automatic analytic continuation of $\mathcal{A}$ -holomorphic functions, which in turn gives a characterization of the domains of $\mathcal{A}$ -holomorphy.

In Section 4 we consider the corresponding analogue of the index (winding number) from the complex plane in this new setting, giving a complete characterization. We remark that some partial examples have been calculated in [4] and [13], but no full description or calculation has been obtained by these authors.

In Section 3 we introduce some auxiliary facts and notions necessary for Section 4. In particular, this includes a homological invariant corresponding to the generalized index introduced in Section 4.

In Section 2 we revisit a small, but key lemma about $\mathcal{A}$ -differentiable 1-forms being d-closed, for which we give a new, but very short proof. Much of our proof strategy relies on this observation.

In Section 1 we give a brief discussion of the basic algebraic preliminaries and fix some definitions needed throughout. We assume however some familiarity

with basic commutative algebra, more precisely with the basic structure theory of Artinian rings.

1. Preliminaries. In the following we fix $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$ to be a local commutative (associative, unital) Artinian ring with a maximal ideal $\mathfrak{m}$ and a residue field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, in particular we have $\mathfrak{m} = \text{nil}(\mathcal{A})$. We will denote by $\chi: \mathcal{A} \rightarrow \mathcal{A}/\mathfrak{m} \cong \mathbb{K}$ the corresponding quotient map. Then $\mathcal{A}$ is automatically a finite-dimensional $\mathbb{K}$ -algebra (i.e. $\mathbb{K}$ is automatically a coefficient field for $\mathcal{A}$), which we can then equip with a submultiplicative norm $\|\cdot\|_{\mathcal{A}}$, for example as inherited along its (faithful) regular representation $\mathcal{A} \hookrightarrow \text{End}_{\mathbb{K}}(\mathcal{A})$, to make $\mathcal{A}$ further a commutative Banach $\mathbb{K}$ -algebra with $\chi: \mathcal{A} \rightarrow \mathbb{K}$ being its unique character. Thus we have a canonical retraction diagram

$$(1.1) \quad \begin{array}{ccc} \mathbb{K} & \xhookrightarrow{\iota} & \mathcal{A} \\ & \searrow \text{id} & \downarrow \chi \\ & & \mathbb{K} \end{array}$$

where ι denotes the corresponding inclusion of scalars, and $\mathcal{A} = \mathbb{K} \oplus \mathfrak{m}$ as $\mathbb{K}$ -vector spaces. Given $Z \in \mathcal{A}$, we will accordingly write $Z = z \oplus X$ with $z = \chi(Z) \in \mathbb{K}$ being the scalar (or spectral¹) part of Z and $X \in \mathfrak{m}$ its nilpotent part. At the same time we also put $[Z] := Z + \mathfrak{m} \in \mathcal{A}/\mathfrak{m}$ to denote the actual equivalence class of Z *before* identification with $\mathbb{K}$, that is, we have $[Z] \subseteq \mathcal{A}$. For the basic structure theory of Artinian rings we refer the reader to any standard text in commutative algebra, for example [1, Ch. 8], and for the theory of coefficients rings and fields to [15]. Every finite-dimensional commutative associative unital $\mathbb{K}$ -algebra, whose all (finitely many) characters are *properly* $\mathbb{K}$ -valued, is a finite direct product of such local Artinian $\mathbb{K}$ -algebras. A complete list of all isomorphism types of local Artinian $\mathbb{K}$ -algebras up to dimension 6 can be found in [9]. However, already in dimension 7 there exist uncountably infinitely many isomorphism types of such algebras, see [16] or [14]. We remark that no explicit classification beyond dimension 7 is known.

Furthermore, we have an isomorphism of groups $\mathcal{A}^{\times} \cong \mathbb{K}^{\times} \times (\mathfrak{m}, +)$ given explicitly by

$$(1.2) \quad \begin{aligned} \mathcal{A}^{\times} &\rightarrow \mathbb{K}^{\times} \times (\mathfrak{m}, +) \\ z \oplus X &\mapsto \left(z, \log \left(1 + \frac{X}{z} \right) \right) \end{aligned}$$

¹Indeed, $\mathcal{A}$ is isomorphic to an upper-triangular algebra with same-entry diagonal, see e.g. [6], with z being the eigenvalue of the operator Z with multiplicity $\dim_{\mathbb{K}} \mathcal{A}$, which justifies the given terminology. In the same vein, we will sometimes call χ a spectral retraction.

as one can check directly. A vector bundle proof of the above isomorphism can be found in [5]. In particular, it follows that $\pi_0(\mathcal{A}^\times) = 1$ if and only if $\mathbb{K} = \mathbb{C}$, i.e. if and only if $\mathcal{A}$ carries a complex structure.

From this point on we fix $\mathbb{K} = \mathbb{C}$. It is not difficult to see that $\mathcal{A}$ carries no *exotic* complex structures, that is, the only solutions of the equation $Z^2 = -1$ in $\mathcal{A}$ come from the inclusion of scalars. We will need the following notations:

Definition 1.1. Let $E \subseteq \mathcal{A}$ (for example, an open subset or the support of a singular 1-cycle).

(1) *Spectral Projection of E :*

$$E^{\text{sp}} := \chi(E) \subseteq \mathbb{C}.$$

(2) *Spectral Closure of E :*

$$\hat{E} := \chi^{-1}(E^{\text{sp}}) = E^{\text{sp}} \times \mathfrak{m}.$$

(3) *Spectral Complement of E :*

$$\check{E} := (E^{\text{sp}})^c \times \mathfrak{m} = (\hat{E})^c$$

Moreover, E will be called *spectrally closed* if $\hat{E} = E$.

2. $\mathcal{A}$ -differentiability and the associated differential 1-form.

Let us first recall the formal definition:

Definition 2.1. Let $U \subseteq \mathcal{A}$ be open, $Z_0 \in U$, and $f: U \rightarrow \mathcal{A}$ a map.

(1) f is called (Frechet) $\mathcal{A}$ -differentiable at Z_0 if there exists $A \in \mathcal{A}$ such that

$$(2.1) \quad f(Z_0 + H) = f(Z_0) + AH + r(H),$$

where $\|r(H)\|_{\mathcal{A}} = o(\|H\|_{\mathcal{A}})$ as $H \rightarrow 0$. In this case $(Df)(Z_0) = f'(Z_0) := A$ is the derivative of f at Z_0 .

(2) f is called $\mathcal{A}$ -holomorphic at Z_0 if there exists a neighbourhood $V \ni Z_0$ such that f is $\mathcal{A}$ -differentiable everywhere on V .

(3) The space of everywhere $\mathcal{A}$ -holomorphic functions on U will be denoted by $\mathcal{O}_{\mathcal{A}}(U)$.

Remarks:

(1) It is not difficult to see that the condition in Eq. 2.1 is the same as f being (totally) differentiable with $\mathcal{A}$ -linear Df , or recasted in differential forms, that $df = f'(Z)dZ$.

- (2) If $f \in \mathcal{O}_{\mathcal{A}}(U)$, then in particular Df is $\mathbb{C}$ -linear, hence f is in fact a holomorphic function of several complex variables, which justifies the choice of terminology. It follows that $\omega := f(Z)dZ \in \Omega_{\text{hol}}^1(U, \mathcal{A})$.
- (3) It is easy to check that $\mathcal{O}_{\mathcal{A}}$ is a sheaf of $\mathcal{A}$ -algebras with a distinguished derivation $f \mapsto f'$.

The main fact we will need is the following:

Lemma 2.2 (Closedness of ω). *If $f \in \mathcal{O}_{\mathcal{A}}(U)$, then $\omega := f(Z)dZ$ is d-closed.*

The usually suggested proof of this fact² is by way of the corresponding (generalized) Cauchy-Riemann equations (see e.g. [17, Thm. 2.2]), which in the case of a general $\mathcal{A}$ can be somewhat tedious of a verification. Of course, a coordinate-free proof is equally possible and in fact very short, which is why we decided to include it here:

Proof. We have $df = f'(Z)dZ$, hence $d\omega = (df) \wedge dZ + f d^2Z = f'(Z)dZ \wedge dZ = 0$. $\square$

3. Admissible points and 1-cycles. Let $U \subseteq \mathcal{A}$ be open and connected, in particular $\pi_0(U) = 1$. Then so are U^{sp} and $\hat{U}$ since χ is continuous and open by virtue of being a projection. We have $\pi_1(\hat{U}) \stackrel{\text{def}}{=} \pi_1(U^{\text{sp}} \times \mathfrak{m}) \cong \pi_1(U^{\text{sp}}) \times \pi_1(\mathfrak{m}) \cong \pi_1(U^{\text{sp}})$, in particular $\pi_1(\mathcal{A}^\times) = \pi_1(\mathcal{A} \setminus \mathfrak{m}) = \pi_1(\mathbb{C}^\times \times \mathfrak{m}) \cong \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$. Moreover, we have implications of retraction diagrams

$$(3.1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\iota} & \mathcal{A} \\ & \searrow \text{id} & \downarrow \chi \\ & & \mathbb{C} \end{array} \Rightarrow \begin{array}{ccc} U^{\text{sp}} & \xrightarrow{\iota} & \hat{U} \\ & \searrow \text{id} & \downarrow \chi \\ & & U^{\text{sp}} \end{array}$$

inducing pushforwards of loops

$$(3.2) \quad \begin{array}{ccccc} \mathcal{L}U & \xrightarrow{\chi\#} & \mathcal{L}U^{\text{sp}} & \xrightarrow{\iota\#} & \mathcal{L}\hat{U} \\ & & \searrow \text{id} & & \downarrow \chi\# \\ & & & & \mathcal{L}U^{\text{sp}} \end{array}$$

with $\gamma^{\text{sp}} := \chi\#(\gamma) \stackrel{\text{def}}{=} \chi \circ \gamma$. More generally, if $G \in \mathbf{Ab}$ and $\Gamma := \sum_{j=1}^m g_j \gamma_j \in Z_1(U, G)$ denotes a (singular) 1-cycle with coefficients in G , we obtain homomorphisms of

²yes, even in most books on Riemann Surfaces;

abelian groups

$$(3.3) \quad \begin{array}{ccccc} Z_1(U, G) & \xrightarrow{\chi_{\#}} & Z_1(U^{\text{sp}}, G) & \xleftarrow{\iota_{\#}} & Z_1(\hat{U}, G) \\ & & \searrow \text{id} & & \downarrow \chi_{\#} \\ & & & & Z_1(U^{\text{sp}}, G) \end{array}$$

where $\Gamma \mapsto \Gamma^{\text{sp}} := \sum_{j=1}^m g_j \gamma_j^{\text{sp}}$, and

$$(3.4) \quad \begin{array}{ccccc} H_1(U, G) & \xrightarrow{\chi_*} & H_1(U^{\text{sp}}, G) & \xleftarrow{\iota_*} & H_1(\hat{U}, G) \\ & & \searrow \text{id} & & \downarrow \chi_* \\ & & & & H_1(U^{\text{sp}}, G) \end{array}$$

where $[\Gamma] \mapsto [\Gamma^{\text{sp}}]$. Note that in general $\iota|_{U^{\text{sp}}}$ does not map into U , but rather into $\hat{U} \supseteq U$. Thus, upon identifying U^{sp} with $\iota(U^{\text{sp}})$ and Γ^{sp} with $\iota_{\#}(\Gamma^{\text{sp}})$ (abuse of notation), Diagram (3.1) encodes a strong deformation retraction $\hat{U} \simeq U^{\text{sp}}$ given explicitly by the homotopy $H: [0, 1] \times \hat{U} \rightarrow \hat{U}$, $H(t, -) := (1-t)\chi \circ \iota + t \text{id}_{\hat{U}}$. Hence, in particular, the isomorphism $\pi_1(\hat{U}) \cong \pi_1(U^{\text{sp}})$ can be also given by a loop homotopy $\mathcal{L}\hat{U} \ni \gamma \simeq \gamma^{\text{sp}} \in \mathcal{L}U^{\text{sp}}$, and we also have $H_1(\hat{U}, G) \cong H_1(U^{\text{sp}}, G)$ with $\Gamma \simeq \Gamma^{\text{sp}}$ in $\hat{U}$, that is, $[\Gamma] = [\Gamma^{\text{sp}}]$ in $H_1(\hat{U}, G)$.

Tautologically, $\text{Sing}(\mathcal{A}) = \mathfrak{m}$ is by definition the singularity of

$$\omega := \frac{dZ}{Z} \in \Omega_{\text{hol}}^1(\mathcal{A}^{\times}, \mathcal{A}).$$

Since $\pi_0(\text{Sing}(\mathcal{A})^c) \stackrel{\text{def}}{=} \pi_0(\mathcal{A}^{\times}) = 1$, certain $\int_{\Gamma} \omega$ can be interpreted as integrating ω around $\text{Sing}(\mathcal{A})$. More generally, for fixed $Z_0 \in \mathcal{A}$ we will consider integrals of the form

$$\omega := \frac{dZ}{Z - Z_0} \in \Omega_{\text{hol}}^1(\mathcal{A} \setminus [Z_0], \mathcal{A}).$$

Thus, given some $\Gamma \in Z_1(U, \mathbb{Z})$, our first order of business is to characterize the set of all $Z_0 \in \mathcal{A}$ such that the above $\int_{\Gamma} \omega$ is well-defined.

Definition 3.1 (Admissible Points).

(1) *Admissible zone of points for a loop: if $\gamma \in \mathcal{L}U$, then*

$$\text{Adm}(\gamma) := \text{Adm}_{\mathcal{A}}(\gamma) := \check{\gamma} \stackrel{\text{def}}{=} \{Z \in \mathcal{A} \mid z \notin \gamma^{\text{sp}}\} \stackrel{\text{def}}{=} (\mathbb{C} \setminus \gamma^{\text{sp}}) \times \mathfrak{m}$$

- (2) *Admissible zone of points for a singular 1-cycle: if $\Gamma := \sum_{j=1}^m g_j \gamma_j \in Z_1(U, G)$, then*

$$\text{Adm}(\Gamma) := \text{Adm}_{\mathcal{A}}(\Gamma) := \check{\Gamma} \stackrel{\text{def}}{=} \bigcap_{j=1}^m \text{Adm}(\gamma_j) \stackrel{\text{def}}{=} (\mathbb{C} \setminus \Gamma^{\text{sp}}) \times \mathfrak{m} \stackrel{\text{def}}{=} \left(\mathbb{C} \setminus \bigcup_{j=1}^m \gamma_j^{\text{sp}} \right) \times \mathfrak{m}$$

- (3) *A subset $E \subseteq \mathcal{A}$ is called admissible for $\Gamma \in Z_1(U, G)$ iff $E \subseteq \text{Adm}(\Gamma)$. Symmetrically, given a subset $E \subseteq \mathcal{A}$, a 1-cycle $\Gamma \in Z_1(U, G)$ is called admissible for E iff E is an admissible set for Γ .*

Remarks.

- (1) Dependence on the ambient space: the definition of $\text{Adm}_{\mathcal{A}}(\Gamma)$ is independent of the topological subspace $\mathcal{A} \supseteq U \supset \Gamma$, but it does implicitly depend on the algebra *structure* $\mathcal{A}$ on the underlying ambient vector space, more precisely on the choice of a $\text{codim}_{\mathbb{C}} 1$ -vector subspace for $\mathfrak{m}$.
- (2) Winding numbers and $\mathcal{A}^\times$: we can characterize $\text{Adm}_{\mathcal{A}}(\Gamma)$ in two different ways:

$$Z_0 \in \text{Adm}_{\mathcal{A}}(\Gamma) \Leftrightarrow z_0 \notin \Gamma^{\text{sp}} \left\{ \begin{array}{l} \Leftrightarrow \text{ind}(\Gamma^{\text{sp}}, z_0) \text{ well-defined in } \mathbb{C} \\ \Leftrightarrow \Gamma - Z_0 \subset \mathcal{A}^\times \end{array} \right.$$

- (3) Existence of admissible points: $\text{Adm}_{\mathcal{A}}(\Gamma)$ is open in $\mathcal{A}$ as a complement of a finite union of continuous preimages of closed sets. If Γ consists of sufficiently nice loops such that each $\mathbb{C} \setminus \gamma_j^{\text{sp}}$ is dense in $\mathbb{C}$, $1 \leq j \leq m$, then $\text{Adm}_{\mathcal{A}}(\Gamma)$ is dense open in $\mathcal{A}$ as itself an intersection of dense open subsets. Thus, if $U \subseteq \mathcal{A}$ is open and Γ is not too pathological in the above sense, then $U \cap \text{Adm}_{\mathcal{A}}(\Gamma) \neq \emptyset$.
- (4) Invariance under translations by the radical: we have $\text{Adm}_{\mathcal{A}}(\Gamma) + \mathfrak{m} = \text{Adm}_{\mathcal{A}}(\Gamma)$, which follows immediately from the fact that $\mathcal{A}^\times + \mathfrak{m} = \mathcal{A}^\times$.

Remark (Spaces of admissible loops and 1-cycles). If $Z_0 \in U$, then the sets of admissible loops and 1-cycles for Z_0 are simply given by $\mathcal{L}(U \setminus [Z_0])$ and $Z_1(U \setminus [Z_0], G)$, respectively.

Definition 3.2 (“Spectral” Homology). *Let $G \in \text{Ab}$.*

- (1) “Spectral” 1-Boundaries:

$$(3.5) \quad B_1^{\text{sp}}(U, G) := \chi_{\#}^{-1}(B_1(U^{\text{sp}}, G)) < B_1(U, G)$$

- (2) First “Spectral” Homology Group:

$$(3.6) \quad H_1^{\text{sp}}(U, G) := Z_1(U, G) / B_1^{\text{sp}}(U, G).$$

- (3) If $\Gamma_1, \Gamma_2 \in Z_1(U, G)$ are spectrally homologous, we will denote this by $\Gamma_1 \approx_{\text{sp}} \Gamma_2$.

Remark. $H_1(U, G)$ is a quotient of $H_1^{\text{sp}}(U, G)$ by the subgroup $B_1(U, G)/B_1^{\text{sp}}(U, G)$.

4. The generalized index.

Definition 4.1 (Generalized Index). Let $\Gamma = \sum_{j=1}^m n_j \gamma_j \in Z_1(U, \mathbb{Z})$. Given $Z_0 \in \text{Adm}_{\mathcal{A}}(\Gamma)$, we define

$$\begin{aligned} \text{Ind}_{\mathcal{A}}(\Gamma, Z_0) &:= \text{Ind}_{\mathcal{A}}^{\mathbb{Z}}(\Gamma, Z_0) \\ &:= \frac{1}{2\pi i} \int_{\Gamma} \frac{dZ}{Z - Z_0} \stackrel{\text{def}}{=} \sum_{j=1}^m n_j \frac{1}{2\pi i} \int_{\gamma_j} \frac{dZ}{Z - Z_0} \stackrel{\text{def}}{=} \sum_{j=1}^m n_j \text{Ind}(\gamma_j, Z_0), \end{aligned}$$

where $\text{Ind}_{\mathcal{A}}^{\mathbb{Z}}$ signifies that Γ has coefficients in $\mathbb{Z} \in \mathbf{Ab}$.

Remarks.

- (1) For example, if $\gamma \in \mathcal{L}\mathbb{C}$ and $z_0 \notin \gamma$, then $\text{Ind}_{\mathbb{C}}^{\mathbb{Z}}(\gamma, z_0) \stackrel{\text{def}}{=} \text{ind}(\gamma, z_0) \in \mathbb{Z}$ is the usual index (winding number) in the complex plane.
- (2) Generically, $\text{Ind}_{\mathcal{A}}(\Gamma, Z_0) \in \mathcal{A}$, but we will shortly see that $\text{Ind}_{\mathcal{A}}^G(\Gamma, Z_0)$ also makes sense for any $G \in \mathbf{Ab}$ and $\Gamma \in Z_1(U, G)$.
- (3) Since $f(Z) := (Z - Z_0)^{-1}$ is $\mathcal{A}$ -holomorphic with $f'(Z) = -(Z - Z_0)^{-2}$, we have that

$$\omega := \frac{dZ}{Z - Z_0} \in \Omega_{\text{hol}}^1(\mathcal{A} \setminus [Z_0], \mathcal{A})$$

is d-closed by Lemma 2.2. Thus, at least in Definition 4.1 it is not necessary to assume any stronger regularity of Γ other than continuity (see e.g. [2, Ch. 1.7])

Lemma 4.2 (Homotopy Invariance of $\text{Ind}_{\mathcal{A}}$). Fix $Z_0 \in \mathcal{A}$ and let $\Gamma_1, \Gamma_2 \in Z_1(\mathcal{A} \setminus [Z_0], \mathbb{Z})$ with $\Gamma_1 \simeq \Gamma_2$. Then $\text{Ind}_{\mathcal{A}}(\Gamma_1, Z_0) = \text{Ind}_{\mathcal{A}}(\Gamma_2, Z_0)$.

Proof. As pointed out above,

$$\omega := \frac{dZ}{Z - Z_0} \in \Omega_{\text{hol}}^1(\mathcal{A} \setminus [Z_0], \mathcal{A})$$

is d-closed. Hence $\int \omega$ is invariant under $\simeq$ within $\mathcal{A} \setminus [Z_0]$. $\square$

Lemma 4.3 (Calculation of $\text{Ind}_{\mathcal{A}}$). *Let $\Gamma \in Z_1(U, \mathbb{Z})$ be $\mathcal{C}_{\text{pw}}^1$ -regular. Then:*

$$(4.1) \quad \forall Z_0 \in \text{Adm}_{\mathcal{A}}(\Gamma): \text{Ind}_{\mathcal{A}}(\Gamma, Z_0) = \text{ind}(\Gamma^{\text{sp}}, z_0) \in \mathbb{Z}.$$

Proof. By $\mathbb{Z}$ -linearity of $\text{Ind}_{\mathcal{A}}(-, Z_0)$ it suffices to verify the claim only for $\gamma \in \mathcal{LU}$. We first prove this for $Z_0 = 0 \in \text{Adm}(\gamma)$. Since $\gamma \simeq \gamma^{\text{sp}}$ in $\mathcal{A}^\times$, we have by Lemma 4.2

$$\text{Ind}_{\mathcal{A}}(\gamma, 0) = \text{Ind}_{\mathcal{A}}(\gamma^{\text{sp}}, 0) = \frac{1}{2\pi i} \int_{\gamma^{\text{sp}}} \frac{dz}{z} \stackrel{\text{def}}{=} \text{ind}(\gamma^{\text{sp}}, 0).$$

Next, substituting $W := Z - Z_0$ and $\delta := \gamma - Z_0$ (shifted curve), we get similarly

$$\begin{aligned} \text{Ind}_{\mathcal{A}}(\gamma, Z_0) &\stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\gamma} \frac{dZ}{Z - Z_0} = \frac{1}{2\pi i} \int_{\delta} \frac{dW}{W} \\ &= \text{ind}(\delta^{\text{sp}}, 0) = \text{ind}(\gamma^{\text{sp}} - z_0, 0) = \text{ind}(\gamma^{\text{sp}}, z_0) \end{aligned}$$

as desired. $\square$

Remarks.

- (1) Since both $\frac{dZ}{Z}$ and $\frac{dz}{z}$ are d-closed, Lemma 4.3 should technically be valid even for continuous Γ in place of $\mathcal{C}_{\text{pw}}^1$ -regularity. Alternatively, $\text{Ind}_{\mathcal{A}}$ can also be *defined* ad hoc entirely topologically for merely continuous Γ by combining Eq. (4.1) with the fact that the analytic index in the complex plane corresponds to the topological degree.
- (2) In particular, we have $\text{Ind}_{\mathcal{A}}(\Gamma, Z) \in \mathcal{A}^\times$ if and only if $\text{ind}(\Gamma^{\text{sp}}, z_0) \neq 0$.
- (3) Let V be a $\mathbb{C}$ -vector space and let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ be a choice of a local $\mathbb{C}$ -algebra structure on V . Then Eq. (4.1) shows that, unsurprisingly, $\text{Ind}_{\mathcal{A}}(\Gamma, Z_0)$ depends on this choice as it is determined by the one-dimensional complex subspace of V spanned by $1_{\mathcal{A}}$ (or equivalently, on the choice of a functional $\chi \in V^*$ as the unique character / spectral projection of $\mathcal{A}$). On the bright side, it is not difficult to show that $\text{Ind}_{\mathcal{A}}$ is at least invariant under algebra endomorphisms.

Corollary 4.4. *The 1-form $X^k dX \in \Omega_{\text{hol}}^1(\mathfrak{m}, \mathfrak{m})$ is d-closed for all $1 \leq k \leq \nu - 2$.*

Proof. Let $\gamma \in \mathcal{LA}^\times$ be arbitrary and write $Z = z \oplus X$ and $\gamma = \gamma^{\text{sp}} \oplus \gamma^{\text{m}}$ accordingly. Using Lemma 4.3, we obtain

$$\begin{aligned}
 \text{ind}(\gamma^{\text{sp}}, 0) &= \text{Ind}_{\mathcal{A}}(\gamma, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dZ}{Z} = \frac{1}{2\pi i} \int_{\gamma} \frac{dz \oplus dX}{z \oplus X} \\
 &= \frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{z} \oplus \sum_{k=1}^{\nu-1} (-1)^k \frac{X^k}{z^{k+1}} \right) (dz \oplus dX) = \\
 &= \left(\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z} \right) \oplus \left(\sum_{k=1}^{\nu-1} (-1)^k X^k \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z^{k+1}} + \sum_{k=0}^{\nu-1} \frac{(-1)^k}{z^{k+1}} \frac{1}{2\pi i} \int_{\gamma} X^k dX \right) = \\
 &= \left(\frac{1}{2\pi i} \int_{\gamma^{\text{sp}}} \frac{dz}{z} \right) \oplus \left(\sum_{k=1}^{\nu-1} (-1)^k X^k \underbrace{\frac{1}{2\pi i} \int_{\gamma^{\text{sp}}} \frac{dz}{z^{k+1}}}_{=0 \text{ as } k \geq 1} + \sum_{k=0}^{\nu-1} \frac{(-1)^k}{z^{k+1}} \frac{1}{2\pi i} \int_{\gamma^{\text{m}}} X^k dX \right) = \\
 &= \text{ind}(\gamma^{\text{sp}}, 0) \oplus \left(\sum_{k=0}^{\nu-1} \frac{(-1)^k}{z^{k+1}} \frac{1}{2\pi i} \int_{\gamma^{\text{m}}} X^k dX \right),
 \end{aligned}$$

hence in particular $\forall 1 \leq k \leq \nu - 2$: $\int_{\gamma^{\text{m}}} X^k dX = 0$ by linear independence of $1/z^{k+1}$, $1 \leq k \leq \nu - 1$. But $\gamma^{\text{m}} \in \mathcal{Lm}$ was arbitrary (say, smooth). This implies the claim. $\square$

Remarks.

- (1) Observe that the above proof is basis-free. Of course, it is possible to prove the claim by a direct calculation of $d(X^k dX)$ after fixing a basis for $\mathfrak{m}$.
- (2) The above proof also shows that Eq. (4.1) is in fact equivalent to all $X^k dX$ being d-closed, $1 \leq k \leq \nu - 2$.

Lemma-Definition 4.5. Let $G \in \mathbf{Ab}$ and $\Gamma = \sum_{j=1}^m g_j \gamma_j \in Z_1(U, G)$. Then

$$(4.2) \quad \forall Z_0 \in \text{Adm}_{\mathcal{A}}(\Gamma): \text{Ind}_{\mathcal{A}}^G(\Gamma, Z_0) := \sum_{j=1}^m g_j \text{Ind}_{\mathcal{A}}^{\mathbb{Z}}(\gamma_j, Z_0) \in G$$

is well-defined.

Proof. By Lemma 4.3 we have $\forall 1 \leq j \leq m$: $\text{Ind}_{\mathcal{A}}(\gamma_j, Z_0) \in \mathbb{Z} \subset \mathcal{A}$. Since G is a $\mathbb{Z}$ -module, the quantity $g_j \text{Ind}_{\mathcal{A}}(\gamma_j, Z_0)$, $1 \leq j \leq m$, is again a well-defined element of G . The claim now follows by linearity. $\square$

Corollary 4.6 (Invariance of $\text{Ind}_{\mathcal{A}}$ under nilpotent translations). *Let $G \in \text{Ab}$ and $\Gamma \in Z_1(U, G)$. If $Z_0 \in \text{Adm}_{\mathcal{A}}(\Gamma)$ and $X_0 \in \mathfrak{m}$, then $\text{Ind}_{\mathcal{A}}^G(\Gamma, Z_0 + X_0) = \text{Ind}_{\mathcal{A}}^G(\Gamma, Z_0)$. Thus we have a well-defined map*

$$(4.3) \quad \begin{aligned} \text{Ind}_{\mathcal{A}}^G(\Gamma, -): \text{Adm}_{\mathcal{A}}(\Gamma)/\mathfrak{m} &\rightarrow G \\ [Z_0] &\mapsto \text{Ind}_{\mathcal{A}}^G(\Gamma, [Z_0]), \end{aligned}$$

where $\text{Adm}_{\mathcal{A}}(\Gamma)/\mathfrak{m}$ simply denotes the image of $\text{Adm}_{\mathcal{A}}(\Gamma)$ under the quotient map $\mathcal{A} \rightarrow \mathcal{A}/\mathfrak{m} \cong \mathbb{C}$ and hence can also be identified with a subset of $\mathbb{C}$. In particular, if $0 \in \text{Adm}_{\mathcal{A}}(\Gamma)$, then $\text{Ind}_{\mathcal{A}}^G(\Gamma, X_0) = \text{Ind}_{\mathcal{A}}^G(\Gamma, 0)$.

Proof. This is immediate from Eq. (4.1). $\square$

Definition 4.7 (Simple Loops in $\mathcal{A}$). *Let $\gamma \in \mathcal{L}\mathcal{A}$ and $Z_0 \in \mathcal{A}$. Then γ is called simple with respect to Z_0 , really $[Z_0]$, if $Z_0 \in \text{Adm}_{\mathcal{A}}(\gamma)$ and $\text{Ind}_{\mathcal{A}}(\gamma, [Z_0]) = 1$.*

Lemma 4.8 (Index is locally constant). *Let $\Gamma \in Z_1(U, G)$. Then the map*

$$\begin{aligned} \text{Ind}_{\mathcal{A}}^G(\Gamma, -): \text{Adm}_{\mathcal{A}}(\Gamma) &\rightarrow G \\ Z &\mapsto \int_{\Gamma} \frac{dW}{W - Z} \end{aligned}$$

is locally constant. In particular, Γ divides $\text{Adm}_{\mathcal{A}}(\Gamma)$ in two types of components: points with zero and non-zero $\text{Ind}_{\mathcal{A}}^G$.

Proof. We first prove the statement for $\Gamma := \gamma \in \mathcal{L}U$. This is a parameter integral with an integrand that is continuously $\mathcal{A}$ -differentiable on the open subset $\text{Adm}_{\mathcal{A}}(\gamma)$. Therefore,

$$\frac{d}{dZ} \text{Ind}_{\mathcal{A}}^{\mathbb{Z}}(\gamma, Z) = \int_{\gamma} \frac{d}{dZ} \left(\frac{1}{W - Z} \right) dW = \int_{\gamma} \frac{dW}{(W - Z)^2} = 0$$

since $(Z - W)^{-1}$ is a global $\mathcal{A}$ -primitive in the variable W of the last integrand. Thus

$$\text{Ind}_{\mathcal{A}}^{\mathbb{Z}}(\gamma, -): \text{Adm}_{\mathcal{A}}(\Gamma) \rightarrow \mathbb{Z}$$

is a locally constant function. Finally, if $\Gamma := \sum_{j=1}^m g_j \gamma_j$ for some $g_j \in G$ and

$\gamma_j \in \mathcal{L}U$, then $\text{Ind}_{\mathcal{A}}^G(\Gamma, -) \stackrel{\text{def}}{=} \sum_{j=1}^m g_j \text{Ind}_{\mathcal{A}}^{\mathbb{Z}}(\gamma_j, -)$ is a linear combination with constant coefficients of locally constant maps, hence locally constant itself. $\square$

Definition 4.9 (Unital Component). *Let $\Gamma \in Z_1(U, \mathbb{Z})$. We put*

$$\mathfrak{C}_{\mathcal{A}}(\Gamma) := \{Z \in \text{Adm}_{\mathcal{A}}(\Gamma) : \text{Ind}_{\mathcal{A}}(\Gamma, Z) \in \mathcal{A}^{\times}\}.$$

Corollary 4.10 (Components of Γ for $G = \mathbb{Z}$). *Let $\Gamma \in Z_1(U, \mathbb{Z})$ and $Z \in \text{Adm}_{\mathcal{A}}(\Gamma)$.*

- (1) *Components of $\text{Ind}_{\mathcal{A}}(\Gamma, -)$: each component of Γ is of the form $C \times \mathfrak{m}$, where $C \subseteq \mathbb{C}$ is a component of the locally constant function $\text{ind}(\Gamma^{\text{sp}}, -)$.*
- (2) *Unital component of $\text{Ind}_{\mathcal{A}}(\Gamma, -)$: we have $\text{Ind}_{\mathcal{A}}(\Gamma, Z_0) \in \mathcal{A}^{\times}$ if and only if the component of Γ^{sp} containing z is bounded in $\mathbb{C}$. In other words, we have $\mathfrak{C}_{\mathcal{A}}(\Gamma) = \{z \in \mathbb{C} \setminus \Gamma^{\text{sp}} : \text{ind}(\Gamma^{\text{sp}}, z) \neq 0\} \times \mathfrak{m}$.*
- (3) *$\mathfrak{C}_{\mathcal{A}}(\Gamma)$ is open, not necessarily connected, and spectrally closed, i.e. $\widehat{\mathfrak{C}_{\mathcal{A}}(\Gamma)} = \mathfrak{C}_{\mathcal{A}}(\Gamma)$.*

Proof. To (1) and (2): Both are immediate from Eq. (4.1) and standard complex analysis.

To (3): $\mathfrak{C}_{\mathcal{A}}(\Gamma)$ is open, as a continuous preimage of $\mathcal{A}^{\times}$, in $\text{Adm}_{\mathcal{A}}(\Gamma)$, which itself is open. Spectral closedness is obvious from the characterization in (2). $\square$

Lemma 4.11 (Invariance of $\text{Ind}_{\mathcal{A}}$ under $\approx_{\text{sp}}$). *Let $\Gamma_1, \Gamma_2 \in Z_1(U, \mathbb{Z})$. Then $\Gamma_1 \approx_{\text{sp}} \Gamma_2$ if and only if $\forall Z \in \check{U} : \text{Ind}_{\mathcal{A}}(\Gamma_1, Z) = \text{Ind}_{\mathcal{A}}(\Gamma_2, Z)$.*

Proof. It suffices to show that $[\Gamma] = 0$ in $H_1^{\text{sp}}(U, \mathbb{Z}) \Leftrightarrow \forall Z \in \check{U} : \text{Ind}_{\mathcal{A}}(\Gamma, Z) = 0$. Indeed, by Eq. (4.1) this is equivalent to $\forall z \in (U^{\text{sp}})^c : \text{ind}(\Gamma^{\text{sp}}, z) = 0 \Leftrightarrow \Gamma^{\text{sp}} \in B_1(U^{\text{sp}}, \mathbb{Z})$. $\square$

5. The generalized homological Cauchy integral formula. We now move on to establish a few versions of Cauchy's Integral Formula for $\mathcal{A}$ -holomorphic functions in gradational order. The first issue to be mindful of is variation of the admissible loops (or 1-cycles) for a predefined point vs. variation of the admissible points for a predefined loop (or 1-cycle). To address this we will heavily rely on homotopy and d-closedness instead of other less conceptual, less powerful, and subsequently more cumbersome approaches.

Proposition 5.1 (Centered Cauchy Integral Formula over $\mathcal{A}$ with Index). *Let $Z_0 \in \mathcal{A}$ be a fixed point, $r > 0$, $\Delta := \Delta_r(Z_0)$, and $f \in \mathcal{O}_{\mathcal{A}}(\Delta)$. Then we have:*

$$(5.1) \quad \forall \gamma \in \mathcal{L}(\Delta \setminus [Z_0]) : \text{Ind}_{\mathcal{A}}(\gamma, Z_0) f(Z_0) = \frac{1}{2\pi i} \int_{\gamma} f(Z) \frac{dZ}{Z - Z_0}.$$

Proof. It suffices to prove the claim for $Z_0 = 0$, otherwise simply consider $\tilde{f}(Z) := f(Z + Z_0)$ and $\tilde{\gamma}(t) := \gamma(t) - Z_0$, $t \in [0, 1]$.³ In particular $\Delta \setminus [Z_0] = \Delta^\times := \Delta \cap \mathcal{A}^\times$. Putting $d := \dim_{\mathbb{C}} \mathfrak{m}$, we have $\Delta^\times \stackrel{\text{def}}{=} \Delta \cap (\mathbb{C}^\times \times \mathbb{C}^d) = \mathbb{D}_r^*(0) \times \mathbb{D}_r(0)^d$. Therefore, $\Delta^\times$ has the same 1-homotopy structure as $\mathcal{A}^\times$, namely $\pi_0(\Delta^\times) = 1$ and $[\mathbb{S}^1, \Delta^\times] \cong [\mathbb{S}^1, \mathbb{D}_r^*(0)] \times [\mathbb{S}^1, \mathbb{D}_r(0)]^d \cong [\mathbb{S}^1, \mathbb{D}_r^*(0)] \cong \mathbb{Z}$, where explicitly $\gamma \simeq \gamma^{\text{sp}} \simeq \gamma_\varepsilon^{\text{sp}}$ with $\gamma_\varepsilon^{\text{sp}}: [0, 1] \rightarrow \mathbb{D}_r^*(0) \times \{0_{\mathfrak{m}}\}$, $t \mapsto \varepsilon e^{2\pi i \mu t}$, for some arbitrary fixed $0 < \varepsilon < r$ and some winding number $\mu \in \mathbb{Z}$. We thus have

$$\frac{\dot{\gamma}_\varepsilon^{\text{sp}}(t)}{\gamma_\varepsilon^{\text{sp}}(t)} = 2\pi i \mu \stackrel{\text{def}}{=} 2\pi i \text{ind}(\gamma_\varepsilon^{\text{sp}}, 0) \stackrel{(4.1)}{=} 2\pi i \text{Ind}_{\mathcal{A}}(\gamma, 0).$$

Since $f(Z) \in \mathcal{O}_{\mathcal{A}}(\Delta)$, it follows that

$$\omega := (f(Z) - f(0)) \frac{dZ}{Z} \in \Omega_{\text{hol}}^1(\Delta^\times, \mathcal{A})$$

is d-closed by Lemma 2.2, and therefore $\int \omega$ is invariant under loop homotopy inside $\Delta^\times$. Thus,

$$\begin{aligned} & \left\| \frac{1}{2\pi i} \int_{\gamma} f(Z) \frac{dZ}{Z} - \text{Ind}_{\mathcal{A}}(\gamma, 0) f(0) \right\|_{\mathcal{A}} \stackrel{\text{def}}{=} \left\| \frac{1}{2\pi i} \int_{\gamma} \omega \right\|_{\mathcal{A}} = \left\| \frac{1}{2\pi i} \int_{\gamma_\varepsilon^{\text{sp}}} \omega \right\|_{\mathcal{A}} = \\ & = \left\| \frac{1}{2\pi i} \int_0^1 \frac{\dot{\gamma}_\varepsilon^{\text{sp}}(t)}{\gamma_\varepsilon^{\text{sp}}(t)} (f(\gamma_\varepsilon^{\text{sp}}(t)) - f(0)) dt \right\|_{\mathcal{A}} = \\ & = \left\| \text{Ind}_{\mathcal{A}}(\gamma, 0) \int_0^1 (f(\gamma_\varepsilon^{\text{sp}}(t)) - f(0)) dt \right\|_{\mathcal{A}} \leq |\mu| \|f - f(0)\|_{\mathcal{A}, \gamma_\varepsilon^{\text{sp}}} \xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned}$$

as desired. $\square$

Remarks.

- (1) A byproduct of the argument used in the proof of Proposition 5.1 is that every point $Z_0 \in \mathcal{A}$ has an open neighbourhood containing an *admissible* loop of an arbitrary index $\text{Ind}_{\mathcal{A}}(\gamma, Z_0) \in \mathbb{Z}$.
- (2) Putting $\mathcal{I}_\gamma := \text{Ind}_{\mathcal{A}}(\gamma, -)$, it follows from Proposition 5.1 that $f|_\gamma$ determines f in all components of the set $\text{Adm}_{\mathcal{A}}(\gamma) \cap \mathcal{I}_\gamma^{-1}((\mathbb{Z} \setminus \{0\}))$, which actually contains stuff outside of Δ . In particular, this set is always unbounded as it contains a copy of $\mathfrak{m}$. One thus obtains an $\mathcal{A}$ -holomorphic continuation of f to an unbounded subset of $\mathcal{A}$. Details follow.

³Of course, replacing $f(Z)$ by $\tilde{f}(Z) := f(Z + Z_0) - f(Z_0)$ instead, one could without loss of generality further take f to be centered, i.e. $f(Z_0) = f(0) = 0$, but unfortunately that would inadvertently obscure parts of the proof.

Proposition 5.2 (Local Cauchy Integral Formula over $\mathcal{A}$). *Let $\Delta \subseteq \mathcal{A}$ be a polydisc and $f \in \mathcal{O}_{\mathcal{A}}(\Delta)$. Then:*

$$(5.2) \quad \forall \gamma \in \mathcal{L}\Delta \quad \forall Z \in \Delta \cap \text{Adm}_{\mathcal{A}}(\gamma): \text{Ind}_{\mathcal{A}}(\gamma, Z)f(Z) = \frac{1}{2\pi i} \int_{\gamma} f(W) \frac{dW}{W - Z}$$

Proof. For notational simplicity we can again assume without loss of generality that $\Delta = \Delta_R(0)$ for some $R > 0$. Fix $\gamma \in \mathcal{L}\Delta$ and recall that $\Delta \cap \text{Adm}_{\mathcal{A}}(\gamma) \neq \emptyset$ (and open) since $\text{Adm}_{\mathcal{A}}(\gamma)$ is open dense in $\mathcal{A}$. Now fix $Z \in \Delta \cap \text{Adm}_{\mathcal{A}}(\gamma)$ arbitrary. Note that by definition $\gamma \in \mathcal{L}(\Delta \setminus [Z])$. There exists a sufficiently small $0 < r < R - |z|$ such that $\Delta_r(Z) \subseteq \Delta$. We claim that there exists $\delta \in \mathcal{L}(\Delta_r(Z) \setminus [Z])$ such that $\gamma \simeq \delta$ (smoothly) in $\Delta \setminus [Z]$. Indeed, putting $d := \dim_{\mathbb{C}} \mathfrak{m}$ as before, we explicitly have $\Delta \setminus [Z] = (\mathbb{D}_R(0) \setminus \{z\}) \times \mathbb{D}_R(0)^d$ and $\Delta_r(Z) \setminus [Z] = \mathbb{D}_r^*(z) \times \prod_{j=1}^d \mathbb{D}_r(\xi_j)$, where $X =: (\xi_1, \dots, \xi_d)$ are the coordinates of the nilpotent part of Z and $\mathbb{D}_r^*(z), \mathbb{D}_r(\xi_j) \subseteq \mathbb{D}_R(0)$, $1 \leq j \leq d$. In particular, again as before, $[\mathbb{S}^1, \Delta \setminus [Z]] \cong [\mathbb{S}^1, \mathbb{D}_R(0) \setminus \{z\}] \cong \mathbb{Z}$ with explicit representatives given this time by $\delta_{\mu}: [0, 1] \rightarrow \Delta \setminus [Z]$, $t \mapsto (z + \varepsilon e^{2\pi i \mu t}, \xi_1, \dots, \xi_d)$, $\mu \in \mathbb{Z}$, for some (any) $0 < \varepsilon < r$. Thus $\exists \mu \in \mathbb{Z}: \gamma \simeq \delta_{\mu}$ (smoothly) in $\Delta \setminus [Z]$ and clearly $\delta_{\mu} \in \mathcal{L}(\Delta_r(Z) \setminus [Z])$ by construction. Since

$$\omega := \frac{f(W)}{W - Z} dW \in \Omega_{\text{hol}}^1(\Delta \setminus [Z], \mathcal{A})$$

is d-closed by Lemma 2.2, we have

$$\frac{1}{2\pi i} \int_{\gamma} \omega = \frac{1}{2\pi i} \int_{\delta_{\mu}} \omega \stackrel{\text{Prop. 5.1}}{=} \text{Ind}_{\mathcal{A}}(\delta_{\mu}, Z)f(Z) \stackrel{\text{Lemma 4.2}}{=} \text{Ind}_{\mathcal{A}}(\gamma, Z)f(Z)$$

as desired. $\square$

Theorem 5.3 (Automatic Analytic Continuation to Spectral Cylinders).

- (1) *Local Automatic Analytic Continuation: Let $\Delta \subseteq \mathcal{A}$ be a polydisc and $f \in \mathcal{O}_{\mathcal{A}}(\Delta)$. Then f admits a unique $\mathcal{A}$ -holomorphic continuation to $\hat{\Delta}$ given by:*

$$(5.3) \quad \forall \gamma \in \mathcal{L}\Delta \quad \forall Z \in \Delta \cap \mathfrak{C}_{\mathcal{A}}(\gamma) \quad \forall X \in \mathfrak{m}: \\ \hat{f}(Z + X) := \frac{1}{2\pi i \text{Ind}_{\mathcal{A}}(\gamma, Z)} \int_{\gamma} f(W) \frac{dW}{W - Z - X}.$$

- (2) *Global Automatic Analytic Continuation: we have a natural isomorphism of $\mathcal{A}$ -algebras*

$$(5.4) \quad \begin{aligned} \mathcal{O}_{\mathcal{A}}(U) &\cong \mathcal{O}_{\mathcal{A}}(\widehat{U}) \\ f &\mapsto \widehat{f}, \end{aligned}$$

whose inverse is given by restriction $g \mapsto g|_U$.

Proof. To (1): To simplify notations we can again assume without loss of generality that $\Delta = \Delta_r(0)$ for some $r > 0$. Fix $\gamma \in \mathcal{L}\Delta$ and recall that by definition $\mathfrak{C}_{\mathcal{A}}(\gamma) \subseteq \text{Adm}_{\mathcal{A}}(\gamma)$ and $\forall Z \in \mathfrak{C}_{\mathcal{A}}(\gamma): \text{Ind}_{\mathcal{A}}(\gamma, Z) \in \mathcal{A}^{\times}$. First, we need to show that $\Delta \cap \mathfrak{C}_{\mathcal{A}}(\gamma) \neq \emptyset$ (it's clearly open). Indeed, since $\gamma \subset \Delta$, we have that $\gamma^{\text{sp}} \in \mathcal{L}(\chi(\Delta)) = \mathcal{L}\mathbb{D}_r(0)$ and $C(\gamma) := \{z \in \mathbb{C} \setminus \gamma^{\text{sp}} : \text{ind}(\gamma^{\text{sp}}, z) \neq 0\} \subset \mathbb{D}_r(0)$ is the *bounded component*⁴ of γ^{sp} , hence explicitly $\Delta \cap \mathfrak{C}_{\mathcal{A}}(\gamma) = C(\gamma) \times \mathbb{D}_r(0)^d$, where as before $d := \dim_{\mathbb{C}} \mathfrak{m} \geq 0$. Next notice that for $X = 0$ the RHS agrees with f on $\Delta \cap \mathfrak{C}_{\mathcal{A}}(\gamma) \subseteq \Delta \cap \text{Adm}_{\mathcal{A}}(\gamma)$ by Proposition 5.2. Furthermore, since $X \in \mathfrak{m}$ and $Z \in \text{Adm}_{\mathcal{A}}(\gamma)$, we have $Z + X \in \text{Adm}_{\mathcal{A}}(\gamma)$ and $\text{Ind}_{\mathcal{A}}(\gamma, Z + X) = \text{Ind}_{\mathcal{A}}(\gamma, Z) \neq 0$, so the RHS remains well-defined. More precisely, if $Z_1 + X_1 = Z_2 + X_2$ for some $Z_1, Z_2 \in \Delta$ and $X_1, X_2 \in \mathfrak{m}$, i.e. $Z_2 = Z_1 + X$ for $X := X_1 - X_2 \in \mathfrak{m}$, then $Z_1 \in \text{Adm}_{\mathcal{A}}(\gamma) \Leftrightarrow Z_2 \in \text{Adm}_{\mathcal{A}}(\gamma)$ and $\text{Ind}_{\mathcal{A}}(\gamma, Z_1 + X_1) = \text{Ind}_{\mathcal{A}}(\gamma, Z_1) = \text{Ind}_{\mathcal{A}}(\gamma, Z_2 - X) = \text{Ind}_{\mathcal{A}}(\gamma, Z_2) = \text{Ind}_{\mathcal{A}}(\gamma, Z_2 + X_2)$, hence

$$\forall \Xi \in (\Delta \cap \mathfrak{C}_{\mathcal{A}}(\gamma)) + \mathfrak{m}: \widehat{f}(\Xi) \stackrel{\text{def}}{=} \frac{1}{2\pi i \text{Ind}_{\mathcal{A}}(\gamma, \Xi)} \int_{\gamma} f(W) \frac{dW}{W - \Xi}$$

is itself well-defined (observe also that $(\Delta \cap \mathfrak{C}_{\mathcal{A}}(\gamma)) + \mathfrak{m}$ is open). The so defined function is now $\mathcal{A}$ -holomorphic since $\text{Ind}_{\mathcal{A}}(\gamma, -)$ is locally constant by Lemma 4.8 (hence in particular $\mathcal{A}$ -differentiable) and one can $\mathcal{A}$ -differentiate under the integral. Since $\Delta \cap \mathfrak{C}_{\mathcal{A}}(\gamma) \neq \emptyset$ and each of the connected components of $\mathfrak{C}_{\mathcal{A}}(\gamma)$ clearly contains a corresponding connected (open) component of $\Delta \cap \mathfrak{C}_{\mathcal{A}}(\gamma)$, it follows now from the identity principle that Eq. (5.3) defines a unique $\mathcal{A}$ -holomorphic continuation of f on each connected component of $\mathfrak{C}_{\mathcal{A}}(\gamma)$.

Next we need to show that this procedure is independent of $\gamma \in \mathcal{L}\Delta$. Let $\delta \in \mathcal{L}\Delta$ be another loop and consider $\mathfrak{C}_{\mathcal{A}}(\delta)$. If $\mathfrak{C}_{\mathcal{A}}(\gamma) \cap \mathfrak{C}_{\mathcal{A}}(\delta) = \emptyset$, there is nothing to show. Suppose now that $\mathfrak{C}_{\mathcal{A}}(\gamma) \cap \mathfrak{C}_{\mathcal{A}}(\delta) \neq \emptyset$. It follows that $\Delta \cap \mathfrak{C}_{\mathcal{A}}(\gamma) \cap \mathfrak{C}_{\mathcal{A}}(\delta) \neq \emptyset$ and again each connected component of $\mathfrak{C}_{\mathcal{A}}(\gamma) \cap \mathfrak{C}_{\mathcal{A}}(\delta)$ contains a connected (open) component of $\Delta \cap \mathfrak{C}_{\mathcal{A}}(\gamma) \cap \mathfrak{C}_{\mathcal{A}}(\delta)$. Thus, if f_{γ} and f_{δ} denote the corresponding analytic continuations of f to $\mathfrak{C}_{\mathcal{A}}(\gamma)$ and $\mathfrak{C}_{\mathcal{A}}(\delta)$,

⁴itself possibly consisting of several connected components for each non-zero integer value of ind ;

respectively, then by the identity principle f_γ and f_δ agree on each connected component of $\mathfrak{C}_\mathcal{A}(\gamma) \cap \mathfrak{C}_\mathcal{A}(\delta)$ as they do so inside Δ . Finally, each $Z \in \Delta$ admits a “local” $\gamma \in \mathcal{L}\Delta$ with $\gamma(t) = Z + \varepsilon e^{2\pi it}$, $t \in [0, 1]$, for some (any) $0 < \varepsilon < r - |z|$, so Δ can be “exhausted” by such $\mathfrak{C}_\mathcal{A}(\gamma)$, that is, $\bigcup_{\gamma \in \mathcal{L}\Delta} \mathfrak{C}_\mathcal{A}(\gamma) = \hat{\Delta}$.

To (2): More generally, the last arguments show that $\hat{U} = \bigcup_{\gamma \in \mathcal{L}U} \mathfrak{C}_\mathcal{A}(\gamma)$ and that $f \in \mathcal{O}_\mathcal{A}(U)$ admits a unique $\mathcal{A}$ -holomorphic continuation $\hat{f} \in \mathcal{O}_\mathcal{A}(\hat{U})$. Thus we get an isomorphism $\mathcal{O}_\mathcal{A}(U) \cong \mathcal{O}_\mathcal{A}(\hat{U})$, $f \mapsto \hat{f}$, of $\mathcal{A}$ -algebras with inverse $g \mapsto g|_U$. $\square$

Corollary 5.4. *The domains of $\mathcal{A}$ -holomorphy are precisely the sets of the form $D \times \mathfrak{m}$, where $D \subseteq \mathbb{C}$ is a domain.*

Proof. This follows from Theorem 5.3 and the fact that every domain in $\mathbb{C}$ is a domain of holomorphy. $\square$

Theorem 5.5 (Spectrally Null-Homological Cauchy Integral Formula over $\mathcal{A}$). *Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$, $f \in \mathcal{O}_\mathcal{A}(U)$, and $\Gamma \in Z_1(U, \mathbb{Z})$. If $[\Gamma] = 0$ in $H_1^{\text{sp}}(U, \mathbb{Z})$, then*

$$(5.5) \quad \forall Z \in U \cap \text{Adm}_\mathcal{A}(\Gamma): \text{Ind}_\mathcal{A}(\Gamma, Z)f(Z) = \frac{1}{2\pi i} \int_\Gamma f(W) \frac{dW}{W - Z}.$$

In particular,

$$(5.6) \quad \int_\Gamma f(W) dW = 0.$$

Proof. As before, we can assume without loss of generality that $Z = 0$ and accordingly $\Gamma \in Z_1(U^\times, \mathbb{Z})$, where $U^\times := U \cap \mathcal{A}^\times$. By Theorem 5.3 we can replace our original function by $f \in \mathcal{O}_\mathcal{A}(\hat{U})$. Now define

$$\omega := f(Z) \frac{dZ}{Z} \in \Omega_{\text{hol}}^1(\hat{U}^\times, \mathcal{A}) \text{ and } \omega^{\text{sp}} := f(z) \frac{dz}{z} \in \Omega_{\text{hol}}^1(U^{\text{sp}} \setminus \{0\}, \mathcal{A}),$$

where note that $\pi_0(U^{\text{sp}}) = 1$ and $\hat{U}^\times \stackrel{\text{def}}{=} (U^{\text{sp}} \setminus \{0\}) \times \mathfrak{m}$. We now have

$$\frac{1}{2\pi i} \int_\Gamma \omega = \frac{1}{2\pi i} \int_{\iota_\# \Gamma^{\text{sp}}} \omega = \frac{1}{2\pi i} \int_{\Gamma^{\text{sp}}} \omega^{\text{sp}} = \text{ind}(\Gamma^{\text{sp}}, 0)f(0) \stackrel{(4.1)}{=} \text{Ind}_\mathcal{A}(\Gamma, 0)f(0)$$

since $\Gamma \simeq \iota_\# \Gamma^{\text{sp}}$ in $\hat{U}^\times$, $d\omega = 0$ by Lemma 2.2, $\Gamma^{\text{sp}} \in B_1(U^{\text{sp}}, \mathbb{Z})$ by hypothesis, and an application of the classical Homological Cauchy Integral Formula in $\mathbb{C}$

in the penultimate step. Finally, Eq. (5.6) follows by applying Eq. (5.5) to $(W - Z)f(W)$. $\square$

Remarks.

- (1) As already mentioned, a similar result has been shown by G. B. Rizza in [10]. Namely, in our language his main statement reads as follows: if $U \subseteq \mathcal{A}$ is a domain, $f \in \mathcal{O}_{\mathcal{A}}(U)$, and $\Gamma \in Z_1(U, \mathbb{Z})$ with $[\Gamma] = 0$ in $H_1(U, \mathbb{Z})$, then

$$\exists N \in \mathbb{N} \forall Z \in U \cap \text{Adm}(\Gamma): f(Z)2\pi iN = \int_{\Gamma} \frac{f(W)}{W - Z} dW$$

In comparison, we only require $[\Gamma^{\text{sp}}] = 0$ in $H_1(U^{\text{sp}}, \mathbb{Z})$.

- (2) Having Theorem 5.5 at our disposal it is clear how to derive a strong Laurent Series Expansion and Residue Theorems. In particular, these are valid under much weaker topological assumptions than the ones stated in [7]. We will state these elsewhere as they belong to $\mathcal{A}$ -meromorphic theory, which is a very long topic on its own.
- (3) Operator Calculus Interpretation: a byproduct of Theorem 5.5 and its proof is that one obtains a holomorphic $\mathcal{A}$ -operator calculus for $\mathcal{A}$ -valued holomorphic functions:

$$(5.7) \quad f(Z) \text{ind}(\Gamma^{\text{sp}}, z) = \frac{1}{2\pi i} \int_{\Gamma^{\text{sp}}} \frac{f(w)}{w - Z} dw.$$

This is a very natural idea in the infinite-dimensional setting as well, but as far as we know no systematic theory of it exists.

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